Lecture 2

Relativistic Hamiltonian Formalisms

and

the Light-Front CQ Model

Basic reviews on relativistic quantum mechanics:

B.D. Keister and W.N. Polyzou: Adv. Nucl. Phys. 20 (1991) 225

F. Coester: Prog. Part. Nucl. Phys. 29 (1992) 1

relativistic formulations of quantum mechanics

\# invariance under Poincaré transformations between inertial coordinate systems:

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}$$

Lorentz transformation
(boost + rotation)

space-time translation

Theorem (due to Wigner): a quantum mechanical model formulated on a Hilbert space preserves probabilities in all inertial coordinate systems if and only if the correspondence between states in different inertial coordinate systems can be realized by a unitary ray representation U(Λ, a) of the Poincaré group.

$$\left|\langle \Psi | \Phi \rangle\right|^2 = \left|\langle \Psi' | \Phi' \rangle\right|^2 \quad \Longrightarrow \quad |\Psi'\rangle = e^{i\phi}\, U(\Lambda, a)\, |\Psi\rangle$$

unitary

group property:

$$U(\Lambda_2, a_2)\, U(\Lambda_1, a_1) = e^{i\phi(2,1)}\, U(\Lambda_2 \Lambda_1, a_2 + \Lambda_2 a_1)$$

$U(\Lambda, a)$ contains the time evolution as a subgroup and therefore its construction is much more complicated than in the non-relativistic case

in non-relativistic quantum mechanics, the time evolution is decoupled from Galilean transformations among different inertial frames

$U(\Lambda, a)$ depends upon the interaction

Poincaré group

10 hermitian generators:

- space-time translations: 4 generators → P_i, H $\longrightarrow$ $P^\mu = \left(\vec{P}, H\right)$
- rotations: 3 generators → J_i
- boosts: 3 generators → K_i

$$J^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} = -J^{\nu\mu}$$

group property $\Rightarrow$

$$\begin{cases} \left[P^\mu, P^\nu\right] = 0 \\ \left[J^{\mu\nu}, P^\rho\right] = i\left(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu\right) \\ \left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho}\right) \end{cases}$$

two subsets of generators: kinematic and dynamic ones

the choice of the subsets defines the form of the dynamics

the subset of kinematic generators can be chosen to leave invariant:

the instant plane	the null plane	the origin
t = const.	$t + z = 0$	$x^{\mu} = 0$
K_i and P^0 are dynamic	$J_{\perp}$ and P^- are dynamic	P^{μ} are dynamic
6 kinematic	7 kinematic	6 kinematic
instant form	front form	point form

the three Dirac's forms

construction of U(Λ, a) within the instant form:

$$R_W^c\left(\Lambda, \frac{P}{M}\right) = L_c^{-1}\left(\frac{P'}{M}\right)\Lambda\, L_c\left(\frac{P}{M}\right)$$

Wigner rotation

P' = Λ P

pure boost from $\vec{0}$ to $\vec{P}$

group property: $U(\Lambda,0) = U\left[L_c\left(\frac{P'}{M}\right)\right] U\left[R_W^c\left(\Lambda,\frac{P}{M}\right)\right] U\left[L_c^{-1}\left(\frac{P}{M}\right)\right]$

$${}_c\langle P'; j'\sigma' | U(\Lambda, a) | P; j\sigma\rangle_c = \delta_{jj'}\,\delta(P' - \Lambda P)\, e^{-iP'\cdot a} \sqrt{\frac{\omega(P')}{\omega(P)}}\, D^j_{\sigma'\sigma}\left[R_W^c\left(\Lambda, \frac{P}{M}\right)\right]$$

mass eigenstates

light-front boosts: $\vec{K}^f = \vec{K} - \hat{z} \times \vec{J}$

$$\begin{cases} K_x^f = K_x + J_y \\ K_y^f = K_y - J_x \\ K_z^f = K_z \end{cases}$$

$$U(\Lambda,0) = U\left[L_f\left(\frac{P'}{M}\right)\right] U\left[R_W^f\left(\Lambda,\frac{P}{M}\right)\right] U\left[L_f^{-1}\left(\frac{P}{M}\right)\right]$$

light-front Wigner rotation

connection between instant and light-front states:

$$\left| P; j\mu \right\rangle_c = \sqrt{\frac{P^+}{P^0}} \sum_{\mu'} D^j_{\mu'\mu}\left[R_{fc}\left(\frac{P}{M}\right)\right] \left| P; j\mu' \right\rangle_f$$

$$R_{fc}\left(\frac{P}{M}\right) = L_f^{-1}\left(\frac{P}{M}\right) L_c\left(\frac{P}{M}\right) = \text{Melosh rotation}$$

light-front spin: $\left(0, \vec{j}_f\right) = \frac{1}{M} L_f^{-1}\left(\frac{P}{M}\right) W$

$$W^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{\nu} J_{\rho\sigma} = \text{Pauli–Lubanski vector}$$

$$\left[j_f^a, j_f^b \right] = i\,\epsilon^{abc}\; j_f^c$$

connection between spins in different forms:

$$\vec{j}_A = L_A^{-1}\left(\frac{P}{M}\right) L_B\left(\frac{P}{M}\right) \vec{j}_B$$

A, B = c, f, p

generalized Melosh rotation
(momentum dependent)

N-particle systems: Bakamjian-Thomas construction

$$M_0 \rightarrow M = M_0 + V$$

the interaction V is inserted in the mass operator and commutes with the free operators $\vec{P}_{free}, i\dfrac{\partial}{\partial \vec{P}_{free}}, \vec{j}_{free}$

the Poincaré generators can be obtained from the free ones simply by substituting the free mass operator with the interacting one

spin compositions in the instant form: $\vec{J}_c = \vec{L} + \vec{S} = \sum_j \left(\vec{\ell}_j + \vec{s}_j \right)$

in the front form: $\vec{J}_f \neq \vec{J}_c$

because of a momentum-dependent composition of spins

$$k_i = L_f^{-1}(P)\, p_i \rightarrow \text{internal momenta} \quad \left(\sum_i k_i = 0 \right)$$

$$\sum_{\lambda_1 \lambda_2} D^{s_1}_{\lambda_1' \lambda_1}\left[R_{fc}(k_1)\right] D^{s_2}_{\lambda_2' \lambda_2}\left[R_{fc}(k_2)\right] \langle s_1 \lambda_1, s_2 \lambda_2 | S M_S \rangle \, | s_1 \lambda_1' \rangle \, | s_2 \lambda_2' \rangle$$

Melosh rotations usual Clebsch-Gordan coefficient

Light-Front Formalism

light-front components of a four-vector: $p = \left(p^{-}, \tilde{p}^{(LF)}\right) \qquad \tilde{p}^{(LF)} = \left(p^{+}, \vec{p}_{\perp}\right)$

$$p^{\pm} = p^{0} \pm \hat{n} \bullet \vec{p} \qquad\qquad p \bullet q = p_{\mu} q^{\mu} = \frac{1}{2}\left(p^{-} q^{+} + q^{-} p^{+}\right) - \vec{p}_{\perp} \bullet \vec{q}_{\perp}$$

unit vector

internal kinematic light-front variables: $\begin{cases} \xi_i = p_i^{+} \big/ P^{+} \\ \vec{k}_{i\perp} = \vec{p}_{i\perp} - \xi_i \vec{P}_{\perp} \end{cases}$

$$P = \text{total momentum} \neq \sum_j p_j \quad \begin{cases} \tilde{P}^{(LF)} = \sum_j \tilde{p}_j^{(LF)} & \text{(independent of the interaction)} \\ P^{-} = \dfrac{M^2 + \left|\vec{P}_{\perp}\right|^2}{P^{+}} \neq \sum_j p_j^{-} \end{cases}$$

$$p_j^{-} = \frac{m_j^2 + \left|\vec{p}_{j\perp}\right|^2}{p_j^{+}}$$

mass operator: $\mathrm{M} = \mathrm{M}_0 + \mathrm{V}$

free mass operator

Poincaré-invariant interaction

$$M_0^2 = \sum_i \frac{m_i^2 + \left|\vec{k}_{i\perp}\right|^2}{\xi_i} \quad \Rightarrow \quad M_0 = \sum_i \sqrt{m_i^2 + \left|\vec{k}_i\right|^2}$$

$$\begin{cases} \vec{k}_i = \left(k_{in}, \vec{k}_{i\perp}\right) \\ k_{in} = \frac{1}{2}\left(\xi_i M_0 - \frac{m_i^2 + \left|\vec{k}_{i\perp}\right|^2}{\xi_i M_0}\right) \end{cases}$$

canonical 3-momentum

Poincaré invariance of V:

- V is invariant under spatial rotations
- V is independent on $\vec{P}$

hadronic wave function $\square_H$: three requirements

1) Eigenstate of the total momentum of the system:

center-of-mass motion factorizes $\Rightarrow$ no spurious effects

$$\square_H = \square^{CM}\left(P^+, \vec{P}_\square\right) \cdot \square_H^{(\mathrm{int.})}\left(\square_i, \vec{k}_{i\square}\right)$$

Note: factorization is preserved by light-front boosts $\left[R_W^f\left(L_f\right) = 1\right]$

2) Relativistic covariance:

$$\Psi^{J\mu}_{(LF)}\left(\xi_i, \vec{k}_{i\perp}; \lambda_i\right) = \sum_{\lambda_i'} \langle \lambda_i \,|\, R^{+} \,|\, \lambda_i' \rangle \, \Psi^{J\mu}_{(c)}\left(\vec{k}_i; \lambda_i'\right)$$

λ_i, λ_i': spin projections

$$R^{+} = \prod_j R^{+}_{\text{Melosh}}\left(\xi_j, \vec{k}_{j\perp}; m_j\right) \underset{\text{spin } 1/2}{\equiv} \prod_j \frac{m_j + \xi_j M_0 - i\vec{\sigma} \cdot \hat{n} \times \vec{k}_{j\perp}}{\sqrt{\left(m_j + \xi_j M_0\right)^2 + \left|\vec{k}_{j\perp}\right|^2}}$$

$\Psi^{J\mu}_{(c)}\left(\vec{k}_i; \lambda_i'\right)$ = eigenfunction of the canonical angular momentum

$$\vec{J}_c = \vec{L} + \vec{S} = \sum_j \left(\vec{\ell}_j + \vec{s}_j\right)$$

3) Eigenstate of the mass operator: $M\,\Psi^{(J\pi)}_{(\mathrm{LF})} = M_H\,\Psi^{(J\pi)}_{(\mathrm{LF})}$

- multiplying by R: $R\,M\,R^{+}\,\Psi^{(J\pi)}_{(\mathrm{c})} = M_H\,R\,R^{+}\,\Psi^{(J\pi)}_{(\mathrm{c})}$

- canonical w.f.: $M_R\,\Psi^{(J\pi)}_{(\mathrm{c})} = M_H\,\Psi^{(J\pi)}_{(\mathrm{c})}$

- Melosh-rotated mass operator: $M_R = R\,M\,R^{+} = R\,M_0\,R^{+} + R\,V\,R^{+}$

M_0 V_R

canonical w.f. satisfies a Schrödinger-like equation:

$$\left[\sum_i \sqrt{m_i^2 + \left|\vec{k}_i\right|^2} + V_R\right]\Psi^{(J\pi)}_{(\mathrm{c})} = M_H\,\Psi^{(J\pi)}_{(\mathrm{c})}$$

OGE model, GBE model, …

the eigenfunctions of CQ potential models able to reproduce hadron masss spectra can be used to construct relativistic CQ wave functions

- reinterpretation of the canonical 3-momentum:

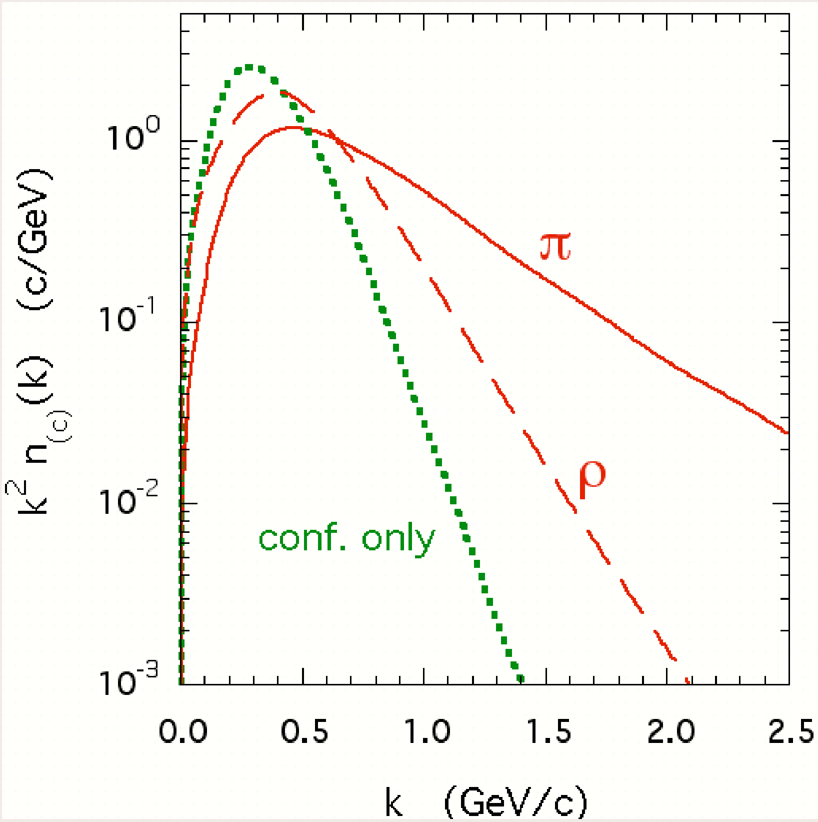
$$\vec{k}_i = \left(k_{in}, \vec{k}_{i\perp}\right), \qquad k_{in} = \frac{1}{2}\left[\xi_i M_0 - \frac{m_i^2 + \left|\vec{k}_{i\perp}\right|^2}{\xi_i M_0}\right]$$

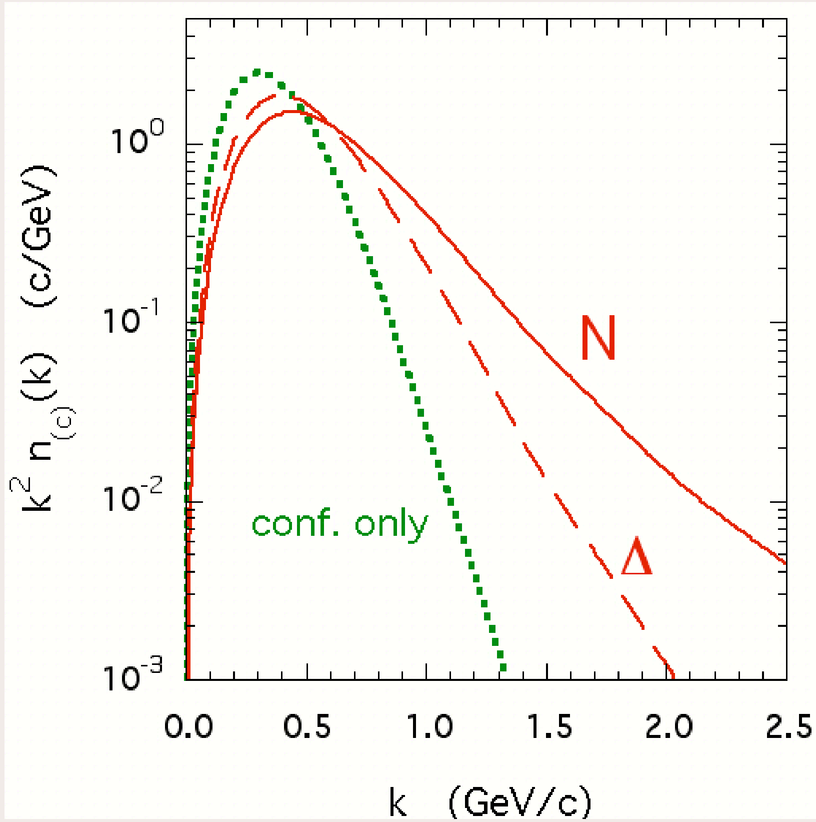
- unitary (Melosh) transformation:

$$\Psi^{J\nu}_{(LF)}\left(\xi_i, \vec{k}_{i\perp}; \nu_i\right) = \sum_{\nu_i'} \langle \nu_i \,|\, R^+ \,|\, \nu_i' \rangle \, \Psi^{J\nu}_{(c)}\left(\vec{k}_i; \nu_i'\right)$$

$$n_{(c)}(k) = \frac{1}{2J+1} \sum_{M=-J}^{J} \sum_{\{\nu_i\}} \int d\Omega_{\vec{k}} \int \left[d\vec{k}_i \right] \delta\left(\vec{k} - \vec{k}_1\right) \left| \Psi_{(c)}^{JM}\left(\vec{k}_i ; \nu_i\right) \right|^2$$

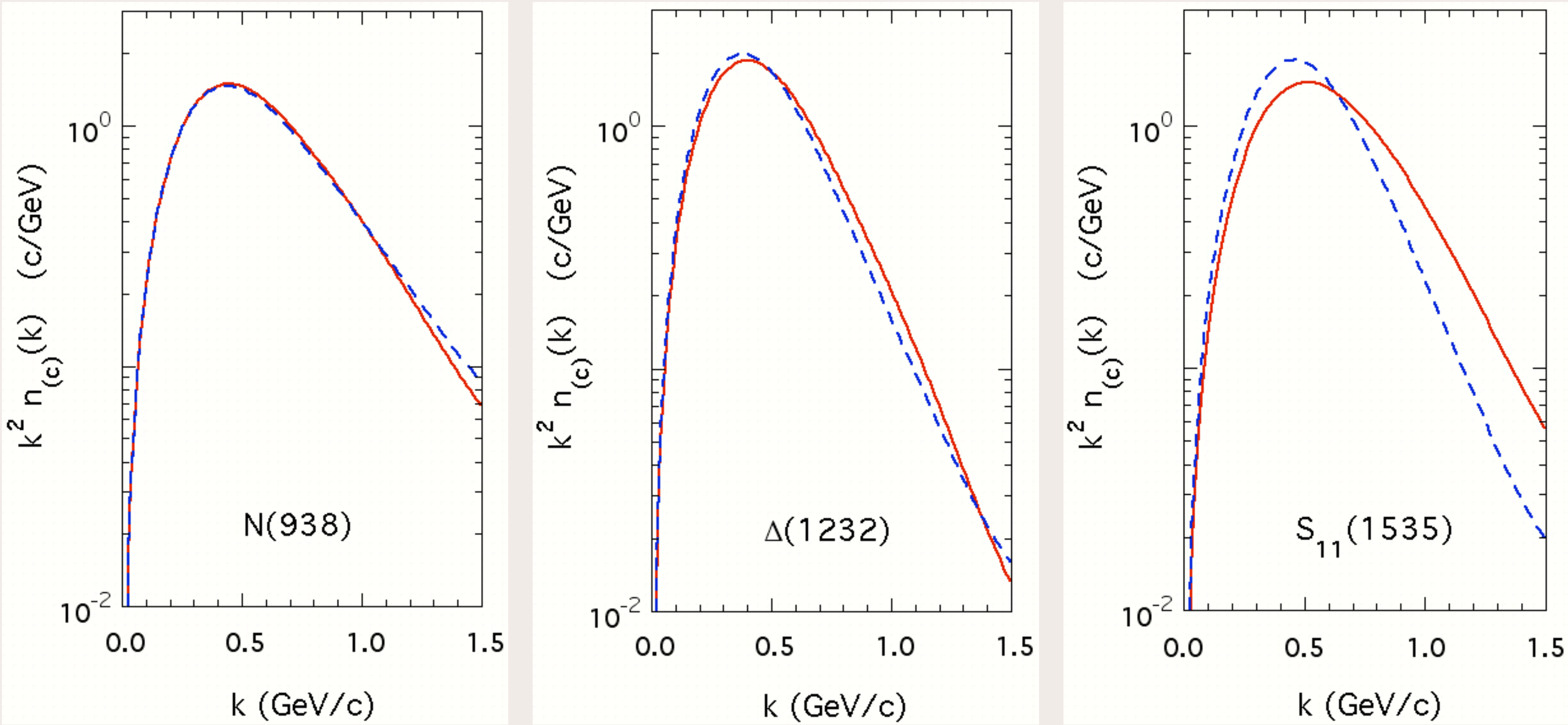
OGE model





large splitting at high momenta due to the spin-spin force [Rome group: ‘94, ‘95]

comparison between OGE and GBE wave functions



OGE

GBE

Electromagnetic Current Operator

Poincaré covariance: $U(\Lambda, a)\, I^{\mu}(x)\, U^{+}(\Lambda, a) = \left[\Lambda^{-1}\right]^{\mu}_{\nu}\, I^{\nu}(\Lambda x + a)$

$$I^{\mu}(x) = e^{-iP\bullet x}\, I^{\mu}(0)\, e^{iP\bullet x}$$

commutation rules:

$$\left[J^{i}, I^{j}(0)\right] = i\varepsilon^{ijk} I^{k}(0), \qquad \left[J^{0}, I^{0}(0)\right] = 0$$

$$\left[K^{i}, I^{j}(0)\right] = -i\delta^{ij} I^{0}(0), \qquad \left[K^{i}, I^{0}(0)\right] = -i\, I^{i}(0)$$

gauge invariance: $\left[P_{\mu}, I^{\mu}(0)\right] = 0$

one-body approximation: $$I^{\mu}(0) \approx \sum_j I_j^{\mu}(0)$$

- to know to what extent the e.m. properties of a system are governed by the e.m. properties of its constituents

problem: the full current operator must contain many-body terms due to the interaction (in any of the forms)

full current → same results in all the forms

one-body approximation → different results in different forms

!!! one important exception: the heavy-quark limit !!!

Which is the best form for the one-body approximation ?

light-front at $q^+ = 0$

in the **instant form** the one-body approximation cannot be formulated in a consistent way

$$\langle P' | I^{\mu}(0) | P \rangle \approx \sum_j \int [d\vec{p}\,'_j] \int [d\vec{p}_j] \langle P' | \{\vec{p}\,'_j\} \rangle \langle \vec{p}\,'_j | I^{\mu}_j(0) | \vec{p}_j \rangle \langle \{\vec{p}_j\} | P \rangle$$

the matrix elements $\langle \vec{p}\,'_j | I^{\mu}_j(0) | \vec{p}_j \rangle$ are related each other by Lorentz boosts which are interaction dependent

in the **point form** the matrix elements of a subset of the components of the one-body approximation play a role in determining the form factors of the system

$q^2 < 0 \Rightarrow q$ along $\hat{x}$ $\qquad q_{\mu} \langle P' | I^{\mu}(0) | P \rangle = 0 \Rightarrow \langle P' | I^{x}(0) | P \rangle = 0$

But point form maximizes the impact of many-body currents

[Simula (‘02), Desplanques et al. (‘02)]

in the front form all the form factors can be determined using the matrix element of only one component of the e.m. current, namely the *“plus”* component $I^+(0)$

- nice feature: $(p_f - p_i)^2 = q^2$ only when $q^+ = 0$

rotational invariance of the charge density: interaction-dependent constraint on the matrix elements of $I^+(0)$

angular condition for $J \geq 1$

the one-body approximation violates the angular condition

the form factors cannot be extracted in a unique way

Elastic Form Factor of the Pion

light-front wave function:

$$\psi_{\pi}^{(LF)} = \sqrt{\frac{A(\xi,\vec{k}_{\perp})}{4\pi}}\, R^{(0)}(\xi,\vec{k}_{\perp})\, w_{\pi}(k) \qquad k = \sqrt{k_n^2 + |\vec{k}_{\perp}|^2}$$

A: normalization factor

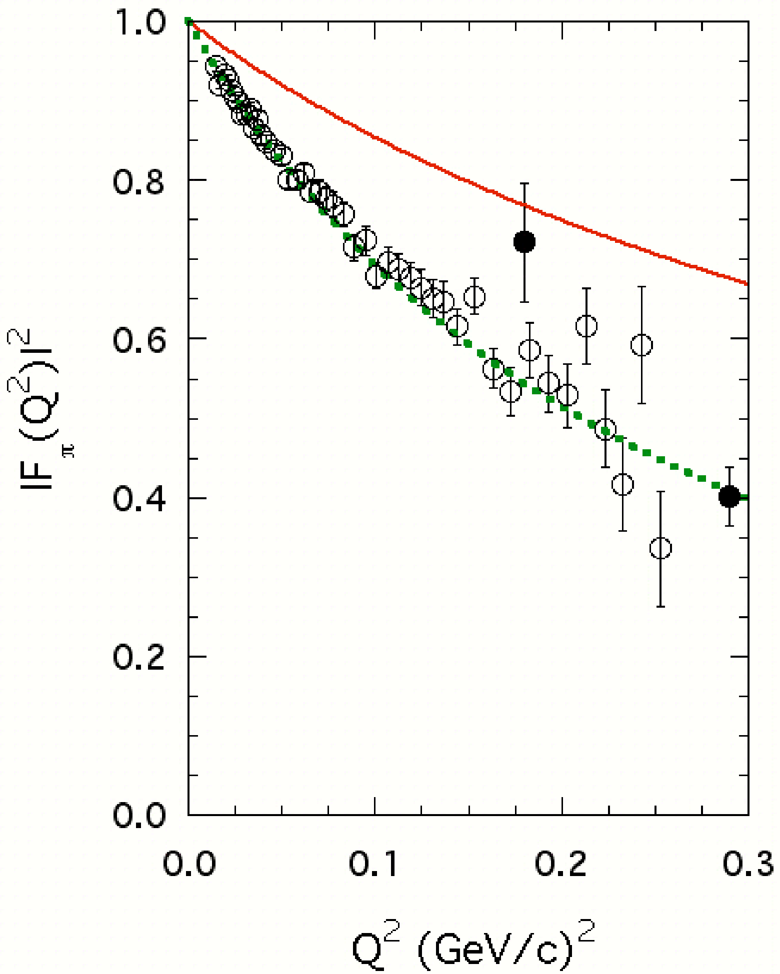
Melosh factor:
$$\left[R^{(0)}(\xi,\vec{k}_{\perp})\right]_{\lambda\bar{\lambda}} = \sum_{\nu\bar{\nu}} \langle \lambda | R_q^{+} | \nu \rangle \langle \bar{\lambda} | R_{\bar{q}}^{+} | \bar{\nu} \rangle \left\langle \frac{1}{2}\nu \frac{1}{2}\bar{\nu} \middle| 00 \right\rangle$$
$$= \frac{1}{\sqrt{2}\, M_0}\, \bar{u}(\tilde{p}_q,\lambda)\, \gamma^5\, v(\tilde{p}_{\bar{q}},\lambda_{\bar{q}})$$

matrix elements of the e.m. current: $\langle P' | I^{\mu}(0) | P \rangle = (P' + P)^{\mu}\, F_{\pi}(Q^2)$

Breit frame at q⁺ = 0: $F_{\pi}(Q^2) = \frac{1}{2P^{+}} \langle P' | I^{+}(0) | P \rangle \qquad Q^2 = -q \bullet q$

point-like CQ's: $I^+(0) = \sum_j e_j \gamma^+$

[Rome group: '94]



full OGE model

conf. only

Amendolia et al.: '84, '86

Brown et al.: '73

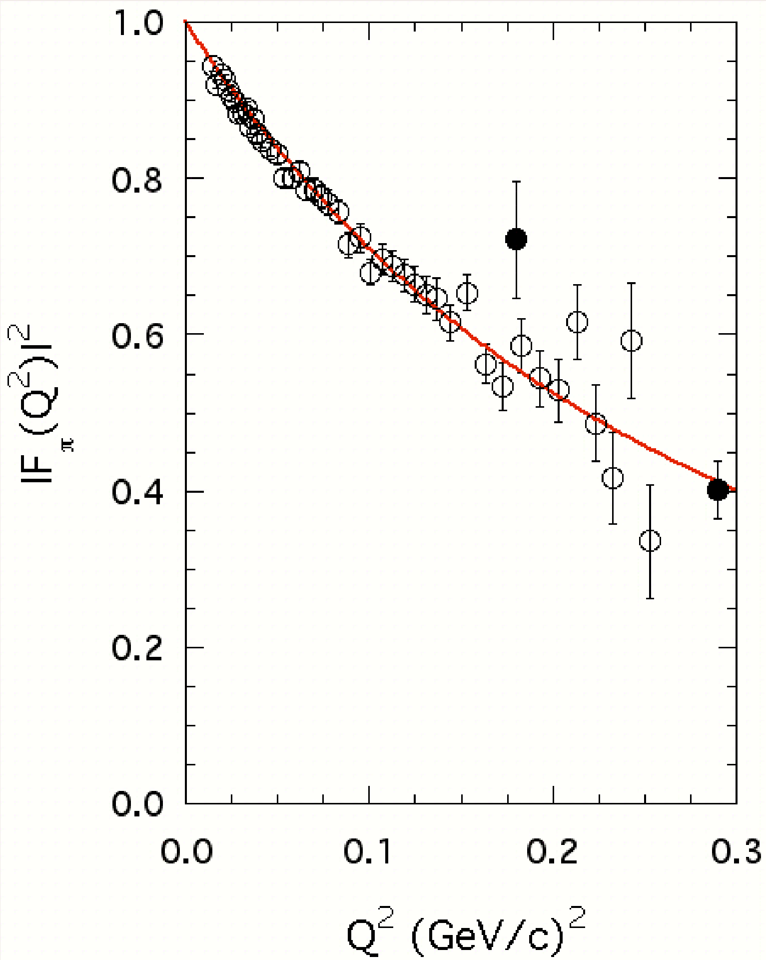
$$\langle r \rangle_\pi = \sqrt{\langle r^2 \rangle_\pi} = 0.660 \pm 0.024 \text{ fm}$$

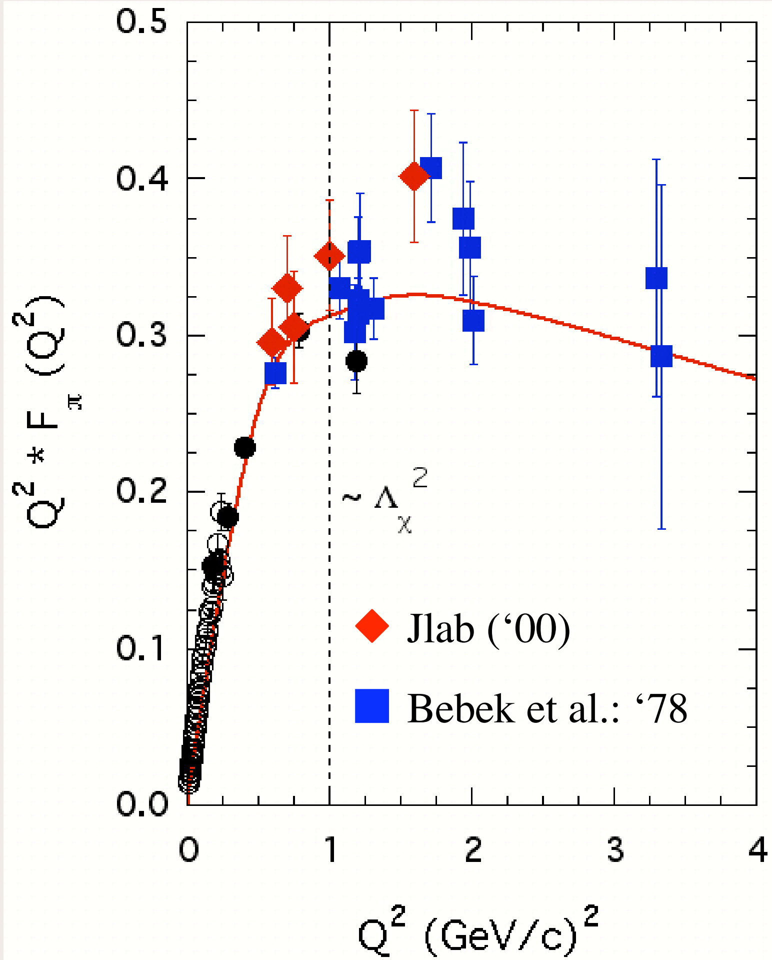
introduce a CQ size

CQ's with a size: $I^+(0) = \sum_j e_j \, f_j(Q^2) \, \Gamma^+$

$$f_U(Q^2) = f_D(Q^2) = \frac{1}{1 + \frac{\langle r^2 \rangle_{CQ} \, Q^2}{6}}$$

- from pion radius: $r_{CQ} = \sqrt{\langle r^2 \rangle_{CQ}} = 0.45 \text{ fm}$





Elastic Form Factors of the □ Meson (spin-1 system)

light-front wave function: $\square_{\square}^{(LF)} = \sqrt{\dfrac{A(\square, \vec{k}_{\square})}{4\square}}\; R^{(1\square)}(\square, \vec{k}_{\square})\; w_{\square}(k)$

Melosh factor: $\left[R^{(1\square)}(\square, \vec{k}_{\square})\right]_{\square\square} = \sum_{\square\square\square\square} \langle \square\square | R_q^{+} | \square \rangle \langle \square\square | R_{\bar{q}}^{+} | \square \rangle \left\langle \frac{1}{2}\square\square\frac{1}{2}\square\square \middle| 1\square \right\rangle$

matrix elements of the e.m. current:

$$\langle 1\square | I^{\square}(0) | 1\square \rangle = \square (P + P\square)^{\square} \left\{ F_1(Q^2)\, e^{*}(\square\square) \bullet e(\square) + \frac{F_2(Q^2)}{2M_V^2}\, e^{*}(\square\square) \bullet q\; e(\square) \bullet q \right\}$$

$$+F_3(Q^2)\left[e^{*\square}(\square\square)\, e(\square) \bullet q \;\square\; e^{\square}(\square)\, e^{*}(\square\square) \bullet q\right]$$

- three form factors

e(□) = polarization 4-vector

- charge form factor: $G_0(Q^2) = F_1(Q^2) + \frac{2\eta}{3}\left[F_1(Q^2) - F_3(Q^2) + (1+\eta)\, F_2(Q^2)\right]$

- magnetic form factor: $G_1(Q^2) = F_3(Q^2)$

$\eta = Q^2/4M_V^2$

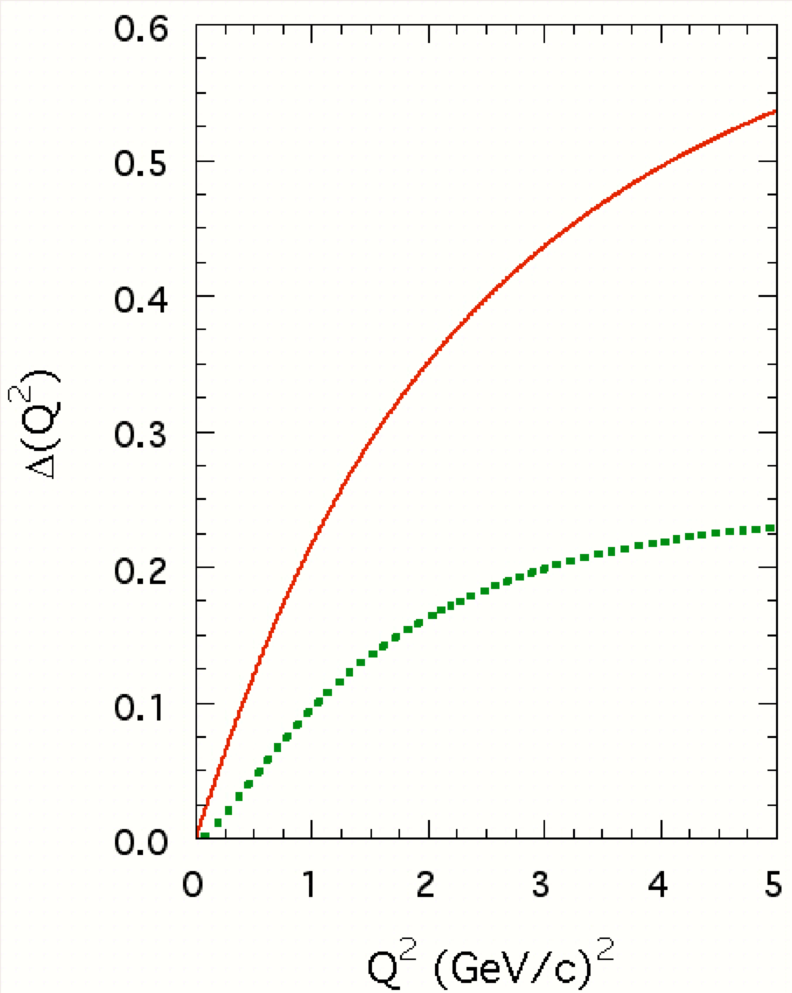
- quadrupole form factor: $G_2(Q^2) = \frac{\sqrt{8}\,\eta}{3}\left[F_1(Q^2) - F_3(Q^2) + (1+\eta)\, F_2(Q^2)\right]$

photon point:
$$\begin{cases} G_0(Q^2=0) = 1 \\ G_1(Q^2=0) = \mu_V = \text{magnetic moment} \\ 3\sqrt{2}\, G_2(Q^2)/Q^2 \;\xrightarrow[Q^2 \to 0]{}\; Q_V = \text{quadrupole moment} \end{cases}$$

hermiticity and time-reversal symmetry: four independent matrix elements of $I^+(0)$

$$I_{11}, I_{1-1}, I_{10}, I_{00} \qquad\qquad I_{\lambda'\lambda} = \langle 1\lambda' |\, I^+(0)\, | 1\lambda\rangle$$

angular condition: $\Delta(Q^2) = (1+2\eta) I_{11}(Q^2) + I_{1-1}(Q^2) - \sqrt{8\eta}\, I_{10}(Q^2) - I_{00}(Q^2) = 0$



point-like CQ's: $I^{+}(0) = \sum_j e_j \gamma^{+}$

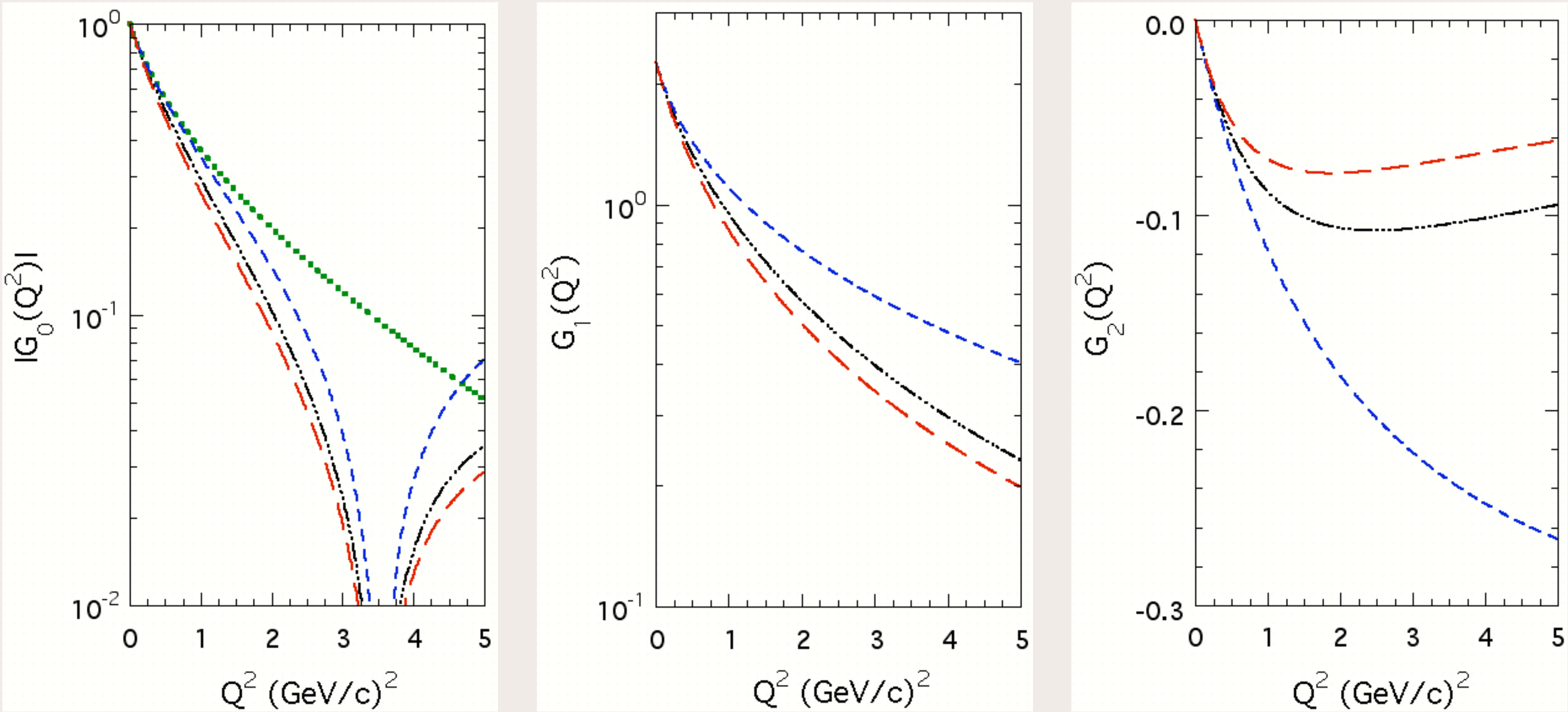
—— full OGE model

.... conf. only

[Rome group: '95]

comparison of prescriptions: BH CCKP FFS GK

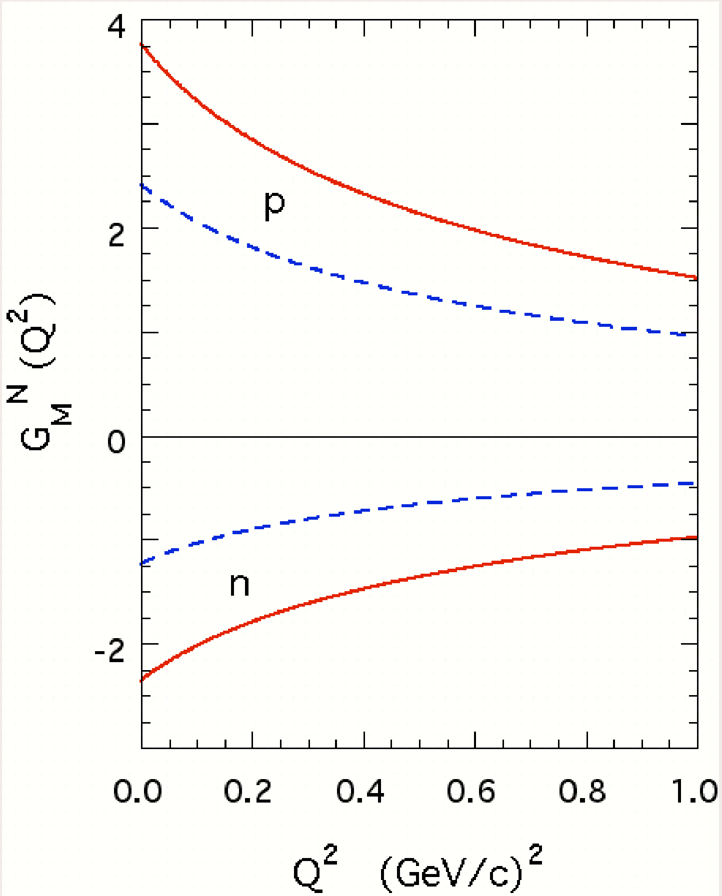
w.f. from OGE model



only magnetic and quadrupole moments are prescription independent

effects of the loss of rotational covariance can be manifest also in systems with J < 1 using different components of the e.m. current

nucleon magnetic form factors:



point-like CQ's: $I^+(0) = \sum_j e_j \gamma^+$

nucleon w.f. from OGE model

from I^+

$$G_M^N(Q^2) = \frac{1}{2}\,\mathrm{Tr}\left[I^+ \left(1 + \frac{2M}{Q}\, i\sigma_y \right) \right]$$

from I^y

$$G_M^N(Q^2) = -\frac{P^+}{Q}\,\mathrm{Tr}\left[I^y\, i\sigma_z \right]$$

relativistic Hamiltonian formalisms: construction of invariant mass operators corresponding to non-relativistic or relativized quark potential models

⇓

reproduction of hadron mass spectra

relativistic wave functions constructed with appropriate spin compositions

light-front form:
- maximum number of kinematic generators
- front boosts form a subgroup
- $(p_i - p_i')^2 = q^2$

loss of rotational covariance !!!

solution in the next lecture